Flow near the Intersection of a Wall and Multiple Dividing Streamlines

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Introduction

STOKES flow streamfunction solutions that satisfy the biharmonic equation

$$\nabla^4 \psi = 0 \tag{1}$$

and contain a single dividing streamline intersection with a plane wall have been considered by several authors.[1-3] Although each puts forth a solution in the polar coordinate form

$$\psi = K_\lambda r^\lambda f_\lambda(\theta) \tag{2}$$

with origin at the intersection, conflicting conclusions are reported. The purposes of this Note are: 1) to clarify the results of these works, in particular the misconceptions presented in Refs. 1 and 3; and 2) to place these results within the context of solutions involving multiple dividing streamline intersections.

Limiting considerations here to Real $(\lambda) > 1$, the solution for the function $f_\lambda(\theta)$ is given by

$$f_\lambda(\theta) = A\cos\lambda\theta + B\sin\lambda\theta + C\cos(\lambda-2)\theta$$

$$+ D\sin(\lambda-2), \ \lambda \neq 2 \tag{3}$$

$$f_2(\theta) = A\cos2\theta + B\sin2\theta + C\theta + D \tag{4}$$

Moffatt[4] considers that the sum of separable solutions may be split into symmetric and antisymmetric flow components such that as $r \to 0$, the solutions with the minimum λ dominate.

The no-slip boundary condition on the velocities

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = 0, \qquad v_\theta = -\frac{\partial\psi}{\partial r} = 0$$

applied at the wall, $\theta = \pm\pi/2$, requires that

$$B = D = 0 \tag{5a}$$

$$A\cos\lambda\pi/2 + C\cos(\lambda-2)\pi/2 = 0 \tag{5b}$$

$$A\lambda\sin\lambda\pi/2 + C(\lambda-2)\sin(\lambda-2)\pi/2 = 0 \tag{5c}$$

for the antisymmetric case, and

$$A = C = 0 \tag{6a}$$

$$B\sin\lambda\pi/2 + D\sin(\lambda-2)\pi/2 = 0 \tag{6b}$$

$$B\lambda\cos\lambda\pi/2 + D(\lambda-2)\cos(\lambda-2)\pi/2 = 0 \tag{6c}$$

for the symmetric case.

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The conditions for the existence of nontrivial solutions are

$$\sin(\lambda-1)\pi = \pm(\lambda-1)\sin\pi = 0 \tag{7}$$

which has as its solution all integers, and only all integers. Equation (7) has been derived for $\lambda > 1$ and $\lambda \neq 2$, but solutions for $\lambda = 0$ and 1 cannot satisfy the no-slip conditions; and solutions for $\lambda = 2$ must satisfy Eq. (4) with $B = C = 0$ and $A = D$, yielding flow parallel to the wall. This solution admits no dividing streamline, in agreement with the conclusion of Dandapat and Gupta,[3] and contradicting the earlier work of Schubert.[1]

Single Dividing Streamline

With $\lambda = 3$, Eq. (5c) gives $3A = C$ for the antisymmetric case, and Eq. (6b) yields $B = D$ for the symmetric case so that

$$f_3^A(\theta) = A(\cos3\theta + 3\cos\theta) \tag{8}$$

$$f_3^S(\theta) = B(\sin3\theta + \sin\theta) \tag{9}$$

where the superscripts indicate the symmetry character of the azimuthal solution. The general streamfunction solution becomes

$$\psi = 4r^3\cos^2\theta(A\cos\theta + B\sin\theta) \tag{10}$$

which, along with the condition for the existence of a dividing streamline

$$\psi(r, \theta_0) = 0 \tag{11}$$

reproduces the Batchelor[2] result

$$\psi = K_3 r^3 \cos^2\theta \sin(\theta - \theta_0) \tag{12}$$

The analysis of Dandapat and Gupta[3] divides the flow into separate regimes on each side of the dividing streamline. The eigenvalues, $\lambda(\theta_0)$, are obtained after imposing Eq. (11), no-slip at the wall, and matching normal and tangential stress along the dividing streamline. One of their results coincides with that presented here as well as that of Moffatt,[4] namely, all real integers $\lambda \geq 3$ are satisfactory eigenvalues, and the streamfunction is analytic throughout. However, their conclusion that there is an additional set of noninteger eigenvalues is erroneous and a direct consequence of using matching conditions which allow discontinuities in streamfunction derivatives across the dividing streamline.

Depending upon the sign of the constant K_3 of Eq. (12), the stagnation point is embedded in either an impinging $(K_3 < 0)$ or separating $(K_3 > 0)$ flow (Fig. 1).

Moreover, for the form of the streamfunction of Eq. (2), the pressure gradient along the wall is given by

$$\frac{\partial p}{\partial r} = \frac{\mu}{r^2}\frac{\partial^2 v_r}{\partial\theta^2} = \mu K_\lambda r^{\lambda-3} f_\lambda''' \tag{13}$$

where μ is the coefficient of viscosity. The wall shear stress is

$$\tau = \frac{\mu}{r}\frac{\partial v_r}{\partial\theta} = \mu K_\lambda r^{\lambda-2} f_\lambda'' \tag{14}$$

For $\lambda = 3$, these expressions become

$$\frac{\partial p}{\partial y} = 6\mu K_3 \sin\theta_0 \tag{15}$$

$$\tau = 2\mu K_3 y \cos\theta_0 \tag{16}$$

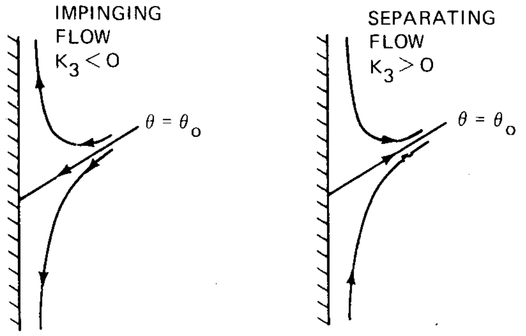

Fig. 1 Flow near a plane wall with a single dividing streamline intersection ($\lambda = 3$).

from which the relationship of Oswatitsch[5] is recovered

$$\cot\theta_0 = 3\frac{\partial\tau/\partial y}{\partial p/\partial y} \tag{17}$$

where y is the vertical coordinate measured positively in the $\theta = \pi/2$ direction.

The adverse pressure gradient at the impingement stagnation point [Eq. (15)] demands that the peak pressure on the wall occurs downstream of the intersection point for all but normal impingement. A similar conclusion was reached by Foss and Kleiss[6] for oblique round jet impingement. For separating flow, the stagnation point is, as expected, also in a region of adverse pressure gradient.

Multiple Dividing Streamlines

It is instructive to put the $\lambda = 3$ results in the context of cases where K_λ vanishes for all λ less than some integer greater than three. To satisfy Eqs. (5a-c) and (6a-c)

$$\left.\begin{aligned} f_\lambda^S &= (\lambda-2)\sin\lambda\theta + \lambda\sin(\lambda-2)\theta \\ f_\lambda^A &= \cos\lambda\theta + \cos(\lambda-2)\theta \end{aligned}\right\}\lambda \text{ even} \tag{18}$$

$$\left.\begin{aligned} f_\lambda^S &= \sin\lambda\theta + \sin(\lambda-2)\theta \\ f_\lambda^A &= (\lambda-2)\cos\lambda\theta + \lambda\cos(\lambda-2)\theta \end{aligned}\right\}\lambda \text{ odd} \tag{19}$$

is required.

The zeros of these equations in the range $-\pi < 2\theta < \pi$ provide acceptable similarity solutions of the Stokes equation and allow for multiple dividing streamline intersections with the wall plane (Figs. 2a and 2b). The general solution at any eigenvalue level is composed of a linear combination of f_λ^S and f_λ^A. For $\lambda = 3$, such an addition of the symmetric normal impingement and the antisymmetric parallel flow streamfunctions produces the oblique impingement case discussed previously. For each eigenvalue, the symmetric and antisymmetric solutions are of either zero wall shear or zero wall pressure gradient, depending upon whether λ is odd or even. With the exception of the $\lambda = 3$ case, the pressure gradient vanishes at the stagnation point. Shear at this point is always zero.

For $\lambda \geq 3$, the zero wall pressure gradient cases contain $(\lambda - 2)$ dividing streamlines, and the zero wall shear cases contain $(\lambda - 3)$ such streamlines. Furthermore, in the vanishing pressure gradient case, these dividing streamlines divide the flow azimuthally into $(\lambda - 1)$ equally spaced regimes. For large values of λ, the dividing streamlines for the zero wall shear solution bisect those regions of the zero wall pressure gradient solution that are bounded by two dividing streamlines. The combination of symmetric and antisymmetric solutions at any level of λ produces a solution with $(\lambda - 2)$ dividing streamlines, suggesting that the zero wall

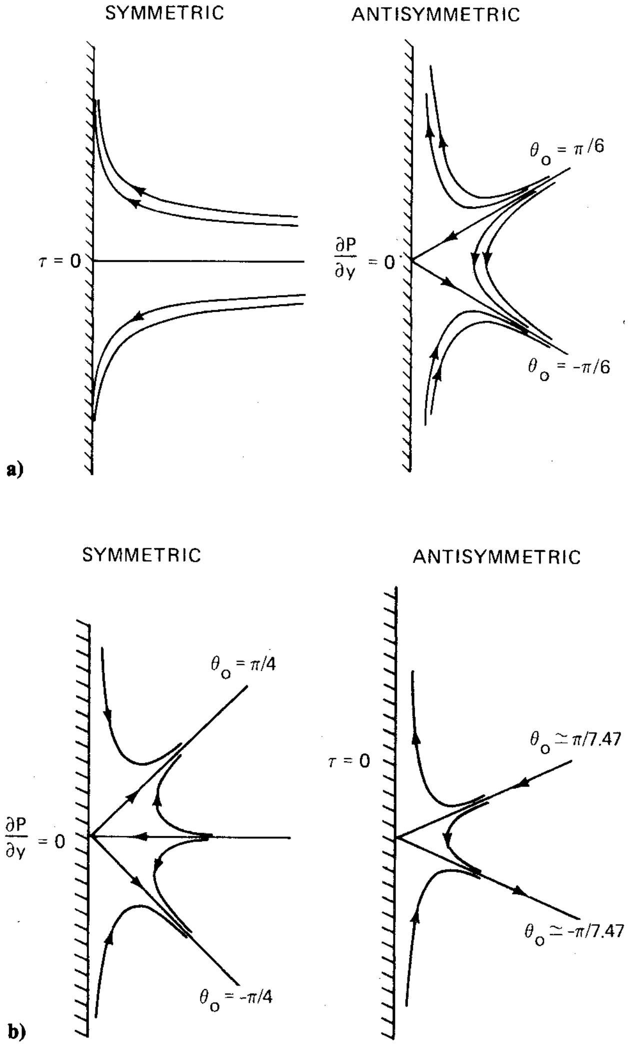

Fig. 2 Multiple dividing streamline intersection solutions; a) $\lambda = 4$, b) $\lambda = 5$.

shear cases, which have one less dividing streamline, are indeed special. Such a combination involves but a single arbitrary constant for purposes of matching the direction of a dividing streamline.

It remains to be seen if, as is so in the case with a single dividing streamline (e.g., Weinbaum[7]), solutions exhibiting multiple dividing streamlines can be matched to realistic outer flows. The case for $\lambda = 4$ could be relevant to the impingement of an oblique jet upon an opposing wall jet and the subsequent upwash formation. However, when multiple dividing streamlines exist near a wall, the possibility always arises that each can intersect the wall separately.

Conclusions

The conditions under which multiple dividing streamlines can intersect a plane wall have been set down in the context of Stokes flow. It is concluded that for separable solutions to the biharmonic governing equation only streamfunctions with integer power radial dependence are allowed (in contrast with Ref. 3). Furthermore, the solution is such that for power laws $\psi = K_\lambda r^\lambda f_\lambda(\theta)$, where $\lambda \geq 3$, as $r \to 0$, there will be $(\lambda - 2)$ dividing streamlines that intersect at the wall, and only one of these can have its direction prescribed arbitrarily.

Ackowledgments

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Approximate Calculation of Vortex Trajectories of Slender Bodies at Incidence

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Nomenclature

a	= body radius
K	= nondimensional constant defining growth of vortex strength with downstream distance
$\tilde{r},r$	= dimensional and nondimensional distance between vortex line and body longitudinal axis, respectively
t	= time
U	= freestream velocity
U_δ	= velocity at boundary-layer edge
W	= undisturbed velocity in the crossflow plane ($=U\sin\alpha$)
x,y,z	= nondimensional coordinates (normalized by a)
$\tilde{x},\tilde{y},\tilde{z}$	= coordinates of vortex lines, defined in Fig. 1
α	= angle of attack
Γ	= vortex strength
Λ	= empirical factor relating vorticity shed from boundary layer to vorticity of the free vortices
ξ	= asymptotic angle of vortices
τ	= characteristic residence time

Introduction

THE aerodynamics of slender bodies at incidence is of fundamental importance to the analysis of high-performance missiles and aircraft. As a result, much work has gone into the understanding of this basically nonlinear situation. (Nielsen[1] has 100 references in a recent review.) A main cause of nonlinearity is the interaction between the shed vorticity and the body flowfield. In the present Note, a simple method for predicting the trajectory of vortices shed on the lee side of slender bodies, not requiring empirical information, is shown. When the angle of attack is less than 20 deg, the vorticity shed rolls up into one or more steady, symmetric pair of vortices on the lee side. In the crossflow plane, this can be pictured as flow around a cylinder with one or more pairs of symmetrically placed, counter-rotating vortices. This case will be examined in the present Note as it highlights all the main points of the model.

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Analysis

Figure 1 shows a typical slender body with one pair of shed vortices. For simplicity, we take the case of a circular cylinder with one pair only, but more vortices or other shapes can be accommodated by a simple generalization.[2,3] Incompressible, steady inviscid (except in the body boundary layer) flow is assumed. Under these circumstances, the flow in the crossflow plane (Fig. 1b) is described by Föppl's[4] classical solution, which can be used to relate the vortex strength and position (in a purely two-dimensional case). Applying the slenderness requirement, the flow in any section A can be approximated by the flow around an infinite cylinder with the local cross section. The relation between vortex strength and position is[5]

$$\Gamma = W\frac{(\tilde{r}^2-a^2)^2(\tilde{r}+a^2)}{\tilde{r}^5} \tag{1}$$

where $W=U\sin\alpha$. Nondimensionalizing by dividing by the radius a, we obtain

$$\frac{\Gamma}{Ua}=\sin\alpha\frac{(r^2-1)^2(r^2+1)}{r^5}=\sin\alpha\left(r-\frac{1}{r}-\frac{1}{r^3}+\frac{1}{r^5}\right) \tag{2}$$

The measured vorticity in the wake is produced by feeding sheets resulting from the separation of boundary layers. The flux of vorticity added to the sheet per unit time and length is[1]

$$\frac{\partial\Gamma}{\partial t}=\Lambda\frac{U_\delta^2}{2} \tag{3}$$

where Λ is a constant (<1) which accounts for loss of vorticity due to mixing, dissipation, etc. For circular cylinders, its value is approximately 0.5.[6] The external speed at the edge of the boundary layer is $U_\delta \approx 2U\sin\alpha$, as separation is assumed to take place at roughly 90 deg from the front stagnation point. Thus, in a fixed-time period τ, the total vorticity shed in a given section is

$$\Gamma=\int_0^\tau 2\Lambda U^2\sin^2\alpha dt \approx 2\Lambda\tau U^2\sin^2\alpha \tag{4}$$

We now apply the slender-body condition again with constant cross section (for simplicity). Each section is then

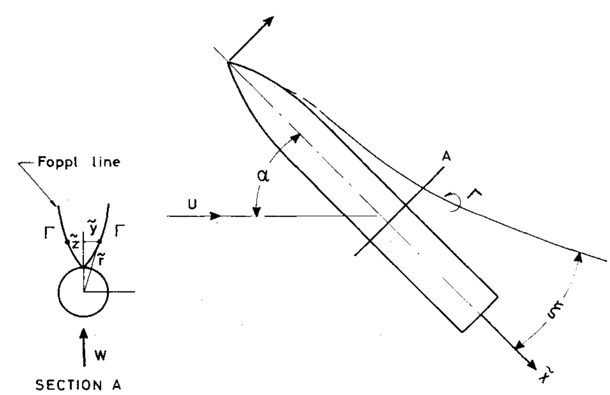

Fig. 1 Schematic description of flowfield and coordinate system.